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The Lie group of automorphisms of a principal bundle

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Abstract. *A convenient structure of Lie group to the entire group Aut P of G-automorphisms of a principal G-bundle without any assumption of compactness on the structure group G or on the base manifold. Its Lie algebra and the exponential map are illustrated. Some relevant principal bundles are discussed having Aut P or its subgroup Gau P of gauge transformations as structure group.*

1. INTRODUCTION

Infinite dimensional Lie groups or infinite dimensional Lie algebras are nowadays understood as unavoidable tools in the formulation of theories of fundamental interactions.

The group *Diff M* of diffeomorphisms of a manifold M is quite familiar since long time to people working in General Relativity. More recently the group *Gau P* of the gauge transformations of a principal bundle (P, p, M, G) gained a similar popularity among people working in Yang-Mills theories.

Convenient smoothness structure for these groups have been proposed along with realizations of their Lie algebras and properties of the exponential map have been investigated ([1, 2, 3] and references therein, [4, 5, 6, 7, 8]).

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When one is interested in coupling gauge fields and gravity, a suitable group to be considered seems to be the group *Aut P*, the group of the automorphisms of *P*. An automorphism of *P* induces a diffeomorphism on the base manifold and, in the case of a trivial *P* or of the bundle *LM* of linear frames, the group *Aut P* is an extension of *Diff M* by *Gau P*.

Although the group *Aut P* is a subgroup of *Diff P* from algebraic point of view, the inherited topology is discrete, hence the Lie structure trivial, in the case of non compact structure group. This difficulty appears already in the group *Gau P* when considered as a subgroup of *Diff P* [9]. This pathology cannot be avoided in the relevant case of frame bundles. As in the case of *Gau P* the way out of the difficulty is to interprete the entire *Aut P* as a space of sections of a suitable fiber bundle. More generally, homomorphisms of principal bundles must be conveniently considered as sections of a naturally constructed fiber bundle.

In Sec 2 we investigate the space $Hom(P, P')$ of homomorphisms of two G-principal bundles and give it a structure of *NLF*-manifold. In particular the smoothness of the evaluation map $Ev : P \times Hom(P, P') \to P'$ is proved. This map - or its restriction to *Aut P* or *Gau P* - is relevant in the cohomological interpretation of anomalies [10]. Universal anomalies are generated by pulling back cohomology classes of classifying spaces via Ev.

The Lie structure of the group *Aut P*, its Lie algebras and the exponential map are illustrated in Sec. 3. *Gau P* is proved to be a splitting subgroup of *Aut P*.

In the last section we discuss some principal bundles. We prove that $Hom(P, P')$ is a *Gau P*-principal bundle with base manifold an open subset of $C^\infty(M, M')$. In the case P' is a universal bundle EG, this fibration can be of interest for the relation between gauge theories and sigma models. Moreover the action of *Gau P* on $Hom(P, EG)$ is linked with the action of *Gau P* on the principal connections via the universal connection theorem. The case of $G = U(k)$ and M a compact manifolds is widely discussed in [11].

As a corollary we have that the exact sequence of *NLF*-Lie groups $1 \to Gau\, P \to Aut\, P \to Diff_{\natural} M \to 1$ is in turn a principal bundle. The group $Diff_{\natural} M$ is an open subgroup of *Diff M* containing the connected component of the identity (and coincides with the entire *Diff M* if *P* is trivial or some other natural bundle).

Moreover, using the results in [14] concerning the action of *Diff M* on the space $Emb(M, M')$ of the embeddings of M into M', we obtain that the space of G-equivariant embedding of P into P' is an *Aut P*-principal bundle.

2. SMOOTHNESS STRUCTURE ON HOM(P, P')

Let $(P, p, M; G)$ and $(P', p', M'; G)$ be two G-principal bundles with M, M' ordinary smooth manifolds and G an ordinary Lie group. We want to endow

$Hom(P, P')$, the set of G-homomorphisms from $(P, p, M; G)$ to $(P', p', M'; G)$, with a convenient smoothness structure.

Our approach is based on the identification of $Hom(P, P')$ with the set of (smooth) sections of a bundle resulting from a natural construction. Such an identification is already known. Using (8.2) of [12] one can identify $Hom(P, P')$ wih the space of the sections of $P[P'] = (P \times_G P', p_{P'}, M)$, the bundle with fiber P' associated to P. Here we prefer to look at this bundle in a slightly different way. The resulting identification is closer to the one well known in the case of morphisms of vector bundles, so we first illustrate our procedure with a short discussion of this latter case.

For two (smooth finite dimensional) vector bundles (E, p, M) and (E', p', M'), a homomorphism from (E, p, M) to (E', p', M') is a fiber preserving (smooth) map $A : E \to E'$ such that the restriction of A to each fiber is a linear map. We denote by $Hom(E, E')$ the space of all homomorphisms from E to E'. By the very definition of homomorphism a (unique, smooth) map $A^\natural : M \to M'$ exists, making the following diagram commutative

$$\begin{array}{ccc} E & \xrightarrow{\;A\;} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M & \xrightarrow{\;A^\natural\;} & M' \end{array}$$

For $x \in M$ and $x' \in M'$, let $L(E_x, E'_{x'})$ be the linear space of the linear mappings from E_x to $E_{x'}$, and put

$$L(E, E') = \coprod_{(x,x')\in M\times M'} L(E_x, E'_{x'}).$$

With the source map $\alpha : L(E, E') \to M$, $\alpha(A_{x,x'}) = z$ and the target map ω : $L(E, E') \to M'$, $\omega(A_{x,x'}) = x'$, $(L(E, E'), \alpha \times \omega, M \times M')$ is a vector bundle. We denote by $L_1(E, E')$ the fiber bundle $(L(E, E'), \alpha, M)$ and by $\Gamma L_1(E, E')$ the space of its sections.

For $A \in Hom(E, E')$ and $x \in M$, let A_x be the restriction of A to the fiber over x. Then a map $\Gamma_A : M \to L(E, E')$ is defined by $\Gamma_A(x) = A_x$, $x \in M$. Clearly, $\Gamma_A \in \Gamma L_1(E, E')$. One easily recognizes that the map $\Gamma : A \to \Gamma_A$ has functorial nature. Actually, Γ is a natural equivalence of the functor $Hom(\, , \,)$, with the functor $\Gamma L_1(\, , \,)$.

It is therefore natural to identify $Hom(E, E')$ with $\Gamma L_1(E, E')$ (see for instance [13]). Using (10.9) of [14] we can endow $Hom(E, E')$ with a structure of C^∞_c-manifold modelled on a suitable nuclear space, inductive limit of nuclear Fréchet spaces (a NLF-space).

Let now come to the principal bundles. We recall that a G-homomorphism from $(P, p, M; G)$ to $(P', p', M'; G)$ is a G-equivariant (smooth) map $\varphi : P \to P'$. In the following $Hom\ (P, P')$ will mean the set of G-homomorphisms from $(P, p, M; G)$ to $(P', p', M'; G)$, $Emb_G(P, P')$ the subset of $Hom(P, P')$ of embeddings, $Iso(P, P')$ the subset of G-isomorphisms and $Aut\,P$ the group $Iso\ (P, P)$.

By the definiton of G-homomorphism a (unique, smooth) map $\varphi^\natural : M \to M'$ exists, making the following diagram commutative

$$\begin{array}{ccc} P & \xrightarrow{\ \varphi\ } & P' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M & \xrightarrow{\ \varphi^\natural\ } & M' \end{array}$$

Thus the map $\natural : Hom(P, P') \to C^\infty\ (M, M')$ can be defined, with associates to each G-homomorphism φ the induces map $\varphi^\natural$. Its image will be denoted by $C^\infty_\natural\ (M, M')$. We also denote the image of $Emb_G(P, P')$ by $Emb_\natural(M, M')$ and the image of $Iso(P, P')$ by $Diff_\natural(M, M')$. The subgroup of G-automorphisms φ such that $\varphi^\natural = id_M$ – often called strong (or vertical) automorphisms – is well known among physicists as the group of gauge transformations. We denote it by $GauP$. We introduce the following maps:

$$Ev : P \times Hom(P, P') \to P',\ Ev(u, \varphi) = \varphi(u);$$

$$Comp : Hom(P', P'') \times Hom(P, P') \to Hom(P, P''),$$

$$Comp(\varphi', \varphi) = \varphi' \circ \varphi;$$

$$Inv : Iso(P, P') \to Iso(P', P),\ Inv(\varphi) = \varphi^{-1}.$$

Then $(Comp(\varphi_2, \varphi_1))^\natural = \varphi_2^\natural \circ \varphi_1^\natural$ and $(Inv(\varphi))^\natural = (\varphi^\natural)^{-1}$.

Consider now the set $Eq(P_x, P'_{x'})$ of all G-equivariant maps from the fiber P_x to the fiber $P'_{x'}$; $Eq(P_x, P'_{x'})$ is in bijection with G and consists of invertible maps. Putting

$$Eq(P, P') = \coprod_{(x, x') \in M \times M'} Eq(P_x, P'_{x'})$$

and defining a source map $\alpha : Eq(P, P') \to M$, $\alpha(\varphi_{x,x'}) = x$, and target map $\omega : Eq(P, P') \to M'$, $\omega(\varphi_{x,x'}) = x'$, we can construct the fibered set $(Eq(P, P'), \alpha \times \omega, M \times M')$. On the other hand, consider the principal bundle $(P \times P', p \times p', M \times M'; G \times G)$, define a left action of its structure group $G \times G$ on G by

$$((a, b), g) \to (a, b)\, g = bga^{-1}$$

and construct the associated bundle $(P \times P')[G] = ((P \times P') \times_{G \times G} G, (p \times p')_G, M \times M')$ whose fibers are diffeomorphic to G.

To discuss the properties of $Eq(P, P')$ it is convenient to introduce the map

$$\eta : P \times P' \times G \to Eq(P, P'),$$

$$\eta(u, u', g)(w) = u'g\tau(u, w) \text{ for } p(w) = p(u).$$

where

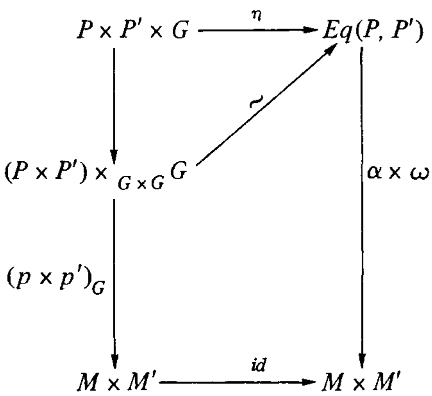
$$\tau : P \times_M P \to G, \text{ defined by } u\tau(u, u') = u'$$

is the translation function of P.

We can state the following theorem.

THEOREM 2.1. *$(Eq(P, P'), \alpha \times \omega, M \times M')$ and $(P \times P')$ $[G]$ are naturally isomorphic as fibered sets on $M \times M'$.*

Proof. The map η factors in a unique bijection from $(P \times P')[G]$ onto $Eq(P, P')$ according to the following diagram



Actually, η is a surjection: for $\varphi \in Eq(P_x, P'_{x'})$, choose $u \in P_x$; then

$$\eta(u, \varphi(u), e)(w) = \varphi(u)\,\tau(u, w) = \varphi(u\tau(u, w)) = \varphi(w)$$

for every $w \in P_x$. Moreover, the fibers of η are exactly the orbits of the joint action of $G \times G$ on $P \times P' \times G$. ∎

By the above theorem $(Eq(P, P'), \alpha \times \omega, M \times M')$ can be identified with the associated bundle $(P \times P')$ $[G]$ and with this identification it becomes a smooth fiber bundle. Therefore also $Eq_1(P, P') = (Eq(P, P'), \alpha, M)$ is a smooth fiber bundle.

Note that the bundles $Eq_1(P, P') = (P \times P') \times_{G\times G} G, pr_1 \circ (p \times p')_G, M)$ and $P[P'] = (P \times_G P', p_{P'}, M)$ are isomorphic. This can be easily seen introducing the maps:

$$\mu : P \times P' \times G \to P \times P', \mu(u, u', g) = (u, u' g)$$

and

$$\nu : P \times P' \to P \times P' \times G, \nu(u, u') = (u, u', e)$$

which are orbit preserving and satisfy the relations

$$\mu \circ \nu(u, u') = (u, u')$$

$$\nu \circ \mu(u, u', g) = (u, u', g)(e, g)$$

so that μ and ν factorize to give an isomorphism of $Eq_1(P, P')$ with $P[P']$.

We will feel free in the following to look at $Eq_1(P, P')$ as $(P \times P')[G]$ or $P[P']$ depending on technical convenience.

It is convenient to introduce the «fiberwise evaluation map»

$$ev : P \times_M Eq(P, P') \to P', \quad ev(u_x, \varphi_x) = \varphi_x(u_x),$$

the «fiberwise composition map»

$$comp : Eq(P', P'') \times_{M'} Eq(P, P') \to Eq(P, P''),$$

$$comp\ (\varphi_{x',x''}, \psi_{x,x'}) = \varphi_{x',x''} \circ \psi_{x,x'}$$

and the «inversion map»

$$inv : Eq(P, P') \to Eq(P', P), inv\ (\varphi_{x,x'}) = (\varphi_{x,x'})^{-1}.$$

LEMMA 2.2. *1) The fiberwise evaluation map ev is a surjective submersion.*
2) The fiberwise composition map comp is smooth.
3) The inversion map inv is smooth.

Proof: 1) Define $ev_M : P \times_M (P \times P' \times G) \to P'$ by $ev_M(u_x, (v_x, u'_{x'}, g)) = u'_{x'}\, g\tau(v_x, u_x)$. Clearly ev_M is smooth and a surjective submersion. Since $u'_{x'} g\tau(v_x, u_x) = \eta(v_x, u'_{x'}, g)(u_x) = ev\ (u_x, \eta(v_x, u'_{x'}, g))$, the following diagram commutes:

$$\begin{array}{ccc} P \times_M (P \times P' \times G) & \xrightarrow{\quad ev_M \quad} & P' \\ \Big\downarrow {\scriptstyle id_P \times_M \eta} & \nearrow {\scriptstyle ev} & \\ P \times_M Eq\,(P, P') & & \end{array}$$

Since $id_P \times_M \eta$ is a surjective submersion, the quotient map ev is smooth and a surjective submersion.

2) The map *comp* is the quotient map of the smooth map $comp_{M'}$, where

$$comp_{M'} : (P' \times P'' \times G) \times_{M'} (P \times P' \times G) \to P \times P'' \times G,$$

$$comp_{M'}\,((u'_{x'}, u''_{x''}, a), (u_x, v'_{x'}, b)) = ((u_x, u''_{x''}, a\tau'(u'_{x'}, v'_{x'})\, b).$$

3) The map inv is the quotient map of the smooth map $inv_{P,P'}$,

$$inv_{P,P'} : P \times P' \times G \to P' \times P \times G,$$

$$inv_{P,P'}(u_x, u'_{x'}, a) = (u'_{x'}, u_x, a^{-1}). \qquad \blacksquare$$

For $\varphi \in Hom\,(P, P')$ and $x \in M$, define $(\Gamma_\varphi)\,(x) \in Eq(P, P')$ as the restriction of φ to P_x. The map $\Gamma_\varphi : x \to (\Gamma_\varphi)(x)$ belongs to $\Gamma\, Eq_1(P, P')$, the space of smooth sections of $Eq_1(P, P')$, since for $u \in P_x$

$$\Gamma_\varphi(x) = \eta(u, \varphi(u), e).$$

Therefore a map

$$\Gamma : Hom\,(P, P') \to \Gamma\, Eq_1(P, P'),\ \ \varphi \to \Gamma_\varphi$$

can be defined.

Analogously, for $s \in \Gamma\, Eq_1(P, P')$, define $\phi_s : P \to P'$ by $\phi_s(u) = s(p(u))(u)$ for $u \in P$. The map ϕ_s is G-equivariant since for $u \in P$ and $g \in G$

$$\phi_s(ug) = s(p(ug))(ug) = s(p(u))\,(u)\,g = \phi_s(u)g,$$

as $s(x)$ is a G-equivariant map for every $x \in M$. By the obvious decomposition $\phi_s = ev \circ (id_P \times s \circ p)$, we obtain that ϕ_s is smooth, hence a homomorphism. Therefore a map $\phi : \Gamma\, Eq_1(P, P') \to Hom\,(P, P')$, $s \to \phi_s$ can be defined. The map ϕ is the inverse of the map Γ since for $w \in p^{-1}(p(u))$

$$\eta(u, \varphi(u), e)(w) = \varphi(u)\tau(u, w)$$

so that for $s = \Gamma_\varphi$ and $u \in P$

$$(\phi_s)(u) = \eta(u, \varphi(u), e)(u) = \varphi(u\tau(u, u)) = \varphi(u).$$

As a consequence of these arguments we can state.

THEOREM 2.3. *The map* Γ *gives a natural equivalence of the functor* $Hom(\, , \,)$ *with the functor* $\Gamma\, Eq_1(\, , \,)$.

We shall therefore identify $Hom(P, P')$ with $\Gamma Eq_1(P, P')$ and give $Hom(P, P')$ a natural C_c^∞ -manifold structure.

THEOREM 2.4. 1) *$Hom(P, P')$ is a NLF-manifold and the map* $\natural : Hom(P, P') \to C_\natural^\infty(M, M')$ *is* C_c^∞ .

2) *$Emb_G(P, P')$ and $Iso(P, P')$ are open submanifolds of $Hom(P, P')$.*

3) *$C_\natural^\infty(M, M')$, $Emb_\natural(M, M')$ and $Diff_\natural(M, M')$ are open submanifolds of $C^\infty(M, M')$. They are closed in $C^\infty(M, M')$, $Emb(M, M')$, $Diff(M, M')$, respectively* .

Proof: We know that $\Gamma\, Eq_1(P, P')$ is a NLF-manifold in the FD-topology [14]; the local model at s is the NLF-space $\Gamma_c\, (s^*\, Ver\, Eq_1(P, P'))$ where *Ver* means the vertical bundle and Γ_c the space of sections with compact support. Therefore $Hom(P, P')$ is a NLF-manifold. Denote by ω_* the pushforward of the sections of $Eq(P, P')$ by means of ω, i.e. $\omega_* : \Gamma\, Eq_1(P, P') \to C_\natural^\infty(M, M')$, $\omega_*(s) = \omega \circ s$. This map is C_c^∞ by (10.14) and (10.10) of [14] and coincides with $\natural$ in the above identification. Hence $\natural$ is a C_c^∞ - map.

The image $C_\natural^\infty(M, M')$ is precisely the set of $f \in C^\infty(M, M')$ such that f^*P' is strongly isomorphic to P. Since f is homotopic to h implies f^*P' is strongly isomorphic to h^*P' (without loss of generality for smooth homotopy), we see that $C_\natural^\infty(M, M')$ (and its complementary set) is a union of arcwise connected components of $C^\infty(M, M')$, so that $C_\natural^\infty(M, M')$ is open and closed in the FD-topology. The images $Emb_\natural(M, M')$ and $Diff_\natural(M, M')$ of $Emb_G(P, P')$ and $Iso(P, P')$ are exactly $Emb(M, M') \cap C_\natural^\infty(M, M')$ and $Diff(M, M') \cap C^\infty(M, M')$, so they are open by (5.3) and (3.7) of [14]. By continuity of the C_c^∞ - map $\natural$, this implies that $Emb_G(P, P')$ and $Iso(P, P')$ are open subsets of $Hom(P, P')$. ■

We stress that the structure of C_c^∞ - manifold given to $Hom(P, P')$ by the above theorem agrees with the structure which $Hom(P, P')$ inherits as a subset of $C^\infty(P, P')$ only in the case G is compact. In this case, $Hom(P, P')$ is a splitting submanifold of $C^\infty(P, P')$. Otherwise, the topology $Hom(P, P')$ inherits from $C^\infty(P, P')$ is a discrete topology. This is rather unpleasant: think of the case of

Hom (E, E'), the space of vector bundle homomorphisms, where the fiber can not be compact. Note that the canonical topology on $L(\mathbf{R}^n, \mathbf{R}^n)$ is the uniform topology induced by operator norm and not the discrete one inherited by $C^\infty(\mathbf{R}^n, \mathbf{R}^n)$.

We remark that $Diff_\natural\, M = Diff_\natural\,(M, M)$ is a NLF-Lie group as open and closed subgroup of the NLF-Lie group *Diff M* and contains therefore the connected component of identity.

To conclude this section, we discuss the C_c^∞-property for the canonical maps *Ev*, *Comp* and *Inv*. It was proved in (11.4) of [14] that the composition law of smooth mappings is a C_c^∞-map only if it is restricted to $C^\infty(M, M'') \times C^\infty_{prop}(M, M')$, where $C^\infty_{prop}\,(M, M')$ denotes the open submanifold of $C^\infty(M, M')$ consisting of proper mappings (see (11.6) of [14]). In dealing with G-homomorphisms of G-principal bundles the same difficulty arises. One is therefore induced to introduce the set of proper G-homomorphisms

$$Hom_{prop}(P, P') = \natural^{-1}(C^\infty_{prop}(M, M'))$$

which, by the above theorem is an open submanifold of $Hom(P, P')$. So we obtain the following theorem.

THEOREM 2.5. *Let $(P, p, M; G)$, $(P', p', M'; G)$ and $(P'', p'', M''; G)$ be G-principal bundles. Then the following maps are C_c^∞:*

1) $Ev : P \times Hom(P, P') \to P'$;

2) $Comp : Hom(P', P'') \times Hom_{prop}(P, P') \to Hom(P, P'')$;

3) $Inv : Iso(P, P') \to Iso(P', P)$.

Proof. We can factorize these canonical maps as follows.

1) By definitions we obtain

$$Ev = ev \circ (id_P \times ev) \circ (Graph\ (p)) \times \Gamma$$

that is

$$Ev(u, \varphi) = ev(u, \Gamma_\varphi(p(u)))$$

for $u \in P$ and $\varphi \in Hom(P', P'')$. By (11.1) and (11.7) of [14] and by 1), Lemma 2.2 we see that Ev in C_c^∞.

2) Let $s = \Gamma_\varphi$, $s' = \Gamma_{\varphi'}$ and $\tilde{s} = \Gamma_{\varphi' \circ \varphi}$ with $\varphi \in Hom_{prop}(P, P')$ and $\varphi' \in Hom(P', P'')$.

Then

$$\tilde{s} = comp \circ ((s' \circ \varphi^\natural) \times s).$$

By Lemma 2.2 and by (10.14) of [14] the map *Comp* is C_c^∞ if $s', s \to s' \circ \omega \circ s$ is C_c^∞. But this is true since $\natural$ $(Hom_{prop}(P, P')) = C^\infty{}_{prop}(M, M') \cap C^\infty_\natural(M, M')$ is an open submanifold of C^∞_{prop} (M, M').

3) Since for $\varphi \in Iso\,(P, P')$

$$\Gamma_{Inv\varphi} = inv \circ \Gamma_\varphi \circ (\varphi^\natural)^{-1},$$

Inv is a C_c^∞-map if the map $s \to inv \circ s \circ (\omega \circ s)^{-1}$ is a C_c^∞-map. By Lemma 2.2, Theorem 2.4 and by (11.11) of [14], the «push-forward» $s \to inv \circ s$ and the map $s \to (\omega \circ s)^{-1}$ are C_c^∞. Their composition is therefore a C_c^∞- map. ∎

3. THE NLF-LIE GROUP AutP AND ITS LIE ALGEBRA

In this section we analyze the structure of the group *AutP* of automorphisms of a principal bundle $(P, p, M; G)$. The results of the above section allow us at first to prove that *AutP* is a NLF-Lie group. Then we give a natural realization of its Lie algebra and its exponential map.

THEOREM 3.1. *AutP is a* NLF-Lie *group and GauP is a closed splitting normal Lie-subgroup of AutP.*

Proof. By Theorem 2.4 we know that *AutP* is open in *Hom*(P, P); hence it inherits the structure of NLF-manifold. The group laws are C_c^∞, since they are respectively the map *Inv* and the restriction of the map *Comp* to open sets. Therefore *AutP* is a NLF-Lie group.

Since *GauP* is the kernel of the homomorphism $\natural$, it is a closed normal subgroup of *AutP*. To prove that it is a splitting Lie-subgroup, we recall that *GauP* is naturally isomorphic to the NLF-Lie group of sections of the associated bundle $P[G] = (P \times_G G, p_G, M)$, where the joint action is

$$(u, g) \to (ua, a^{-1} ga) \quad for \quad a \in G$$

and its Lie algebra is identified with $\Gamma_c P[\mathrm{g}]$, where $P[\mathrm{g}] = (P \times_G \mathrm{g}, p_G, M)$ denotes the associated bundle with the *Ad* action of G on its Lie algebra g.

Consider the embedding $\iota_P : P \times G \to P \times P \times G$, $(u, g) \to (u, u, g)$. It is immediate to verify that a unique ι exists making the following diagram commutative

$$\begin{array}{ccc}
P \times G & \xrightarrow{\iota_P} & P \times P \times G \\
\Big\downarrow q & & \Big\downarrow \eta \\
P \times_G G & \xrightarrow{\iota} & (P \times P) \times_{G \times G} G = Eq(P, P) \\
\Big\downarrow & & \Big\downarrow \alpha \\
M & \xrightarrow{id} & M
\end{array}$$

It follows by the above diagram that ι is smooth and open on its image. By direct inspection one easily checks that ι is injective. Recall that TP is turn a principal bundle with structure group TG. Hence the tangent map of ι is injective for the same reason as ι. Thus ι is an embedding, $\Gamma P[G]$ is a splitting submanifold of $\Gamma\, Eq_1(P, P)$ and the C_c^∞ inclusion map is simply ι_*, $\iota_*(s) = \iota \circ s$ (see (10.8) and (10.10) of [14]).

The range of ι_* is contained in $AutP$, wich is an open submanifold of ΓEq_1 (P, P) and ι_* is a group homomorphism. ■

For a NLF-Lie group $\mathscr{G}$, the Lie algebra $\mathscr{L}(\mathscr{G})$ of $\mathscr{G}$ is defined as usual as the Lie algebra of left invariant vector fields on $\mathscr{G}$. Lie brackets can be defined by means of the identification of tangent vectors with continuous derivations, identification which remains true in the setting of NLF-manifolds [6]. As a topological space, $\mathscr{L}(\mathscr{G})$ is identified with the tangent space at the identity of $\mathscr{G}$.

We known that $AutP$ is an open subset of the NLF-manifold $Hom(P, P) \simeq \Gamma Eq_1(P, P)$. The tangent space at $s \in \Gamma Eq_1(P, P)$ is the NLF-space $\Gamma_c(s^*\, Ver\, Eq_1(P, P))$ of sections with compact support of the pullback via s of the vertical bundle of $Eq_1(P, P)$. Hence the tangent space at the identity of $AutP$ is the NLF-space $\Gamma_c(\mathrm{e}^*\, Ver\, Eq_1(P, P))$, where e denotes the identity section of $Eq_1(P, P)$, $\mathrm{e}(x) = id_x$, for $x \in M$.

But another realization of $\mathscr{L}(AutP)$ is expected. In fact, it is well known that $\mathscr{L}(Diff\ M)$ is naturally anti-isomorphic with the Lie algebra $\mathscr{X}_c(M)$ of all vector fields on M with compact support [6]. Analogously, we well prove that $\mathscr{L}(AutP)$ is naturally anti-isomorphic with a suitable subalgebra of the

Lie algebra $\mathscr{X}_G(P)$ of all G-invariant vector fields on TP.

To this purpose we need some preliminaries. It is well known that $(TP, Tp, TM; TG)$ is a TG-principal bundle with the tangent action. One can therefore construct the associated bundle $(TP \times TP')\,[TG]$ with respect to the tangent action of $TG \times TG$. The related action is just the tangent action of the joint action of $G \times G$ on $P \times P' \times G$, defined by

$$(\xi_u, \xi'_{u'}, k_g)\,(h_a, h'_b) = (\xi_u . h_a, \xi'_{u'} . h'_b, (h'_b)^{-1} . k_g . h_a).$$

The bundles $T((P \times P')\,[G])$ and $(TP \times TP')\,[TG]$ are naturally isomorphic as bundles on TM. Let $\tau_P, \tau_{P'}, \tau_M, \tau_G$ and $\tilde{\tau}$ denote the projections of the tangent bundles TP, TP', TM, TG and $(TP \times TP')\,[TG]$ over $(P \times P')\,[G]$, respectively. By the diagram

$$\begin{array}{ccc}
TP \times TP' \times TG & \xrightarrow{\tau_P \times \tau_{P'} \times \tau_G} & P \times P' \times G \\
\Big\downarrow T\eta & & \Big\downarrow \eta \\
T((P \times P')\,[G]) = (TP \times TP')[TG] & \xrightarrow{\tilde{\tau}} & (P \times P')[G] = Eq_1(P, P') \\
\Big\downarrow T\alpha & & \Big\downarrow \alpha \\
TM & \xrightarrow{\tau_M} & M
\end{array}$$

we see that $T\alpha(T\eta(\xi_u, \xi'_{u'}, k_g)) = Tp(\xi_u)$ for $(\xi_u, \xi'_{u'}, k_g) \in TP \times TP' \times TG$ so that

$$ker\ T\alpha = Ver\ Eq_1(P, P') = T\eta((Ver\ P) \times TP' \times TG).$$

LEMMA 3.2. *Let $u \in P$ and $u' \in P'$. The map*

$$T\eta(0_u, \cdot, 0_e) \quad : T_{u'}P' \to (Ver\ Eq_1(P, P'))_{\eta(u,u',e)}$$

is a linear isomorphism .

Proof. Injectivity is clear by inspection of the formula. Surjectivity follows by counting dimensions. ∎

Note that $\eta(u, u, e) = \mathbf{e}(x)$ if $p(u) = x$, where $\mathbf{e}$ is the identity section of $Eq_1(P, P)$. As a first consequence of the above lemma, the fiber on x of the pullback $\mathbf{e}^*Ver\ Eq_1(P, P)$ is precisely $\{T\eta(0_u, \xi_u, 0_e), \xi_u \in T_uP \text{ for } p(u) = x\}$.

Let $\xi \in \mathscr{X}_G(P)$. Since for any $g \in G$, $T\eta(0_{ug}, \xi(ug), 0_e) = T\eta(0_u, \xi(u), 0_e)$ and $\tilde{\tau}(T\eta(0_u, \xi(u), 0_e)) = \eta(u, u, e)$, we see that a map

$$I : \mathscr{X}_G(P) \to \Gamma(\mathbf{e}^*\ Ver\ Eq_1(P, P))$$

$$I(\xi)(x) = T\eta(0_u, \xi_u, 0_e) \ \textit{for} \ p(u) = x$$

is well defined.

LEMMA 3.3. *The map* $I : \mathscr{X}_G(P) \to \Gamma(\mathbf{e}^*\ Ver\ Eq_1(P, P))$ *is a natural linear isomorphism. Moreover, for* $\xi \in \mathscr{X}_G(P)$

$$supp(I(\xi)) = p(supp(\xi)).$$

Proof. Clearly I is a linear map and $\xi(u) = 0$ if and only if $I(\xi)\ (p(u)) = 0$. Therefore I is an injection and $supp(I(\xi)) = p(supp(\xi))$. To prove surjectivity of I, let $\sigma \in \Gamma(\mathbf{e}^*\ Ver\ Eq_1(P, P))$. By Lemma 3.2, for every $u \in P$ with $p(u) = x$, a unique $\xi_u \in T_uP$ exists such that

$$\sigma(x) = T\eta(0_u, \xi_u, 0_e).$$

By means of local charts one easily proves that $\xi : P \to TP$, $\xi(u) = \xi_u$ is a smooth section. By uniqueness property one obtains that $\xi \in \mathscr{X}_G(P)$, since for $u \in P$ and $g \in G$

$$T\eta(0_{ug}, \xi_u . g, 0_e) = T\eta(0_u, \xi_u, 0_e) = \sigma(p(u)).$$

Clearly, $I(\xi) = \sigma$. ∎

As a consequence of Lemma 3.3, the tangent space at $\mathbf{e}$ of $AutP$, i.e. the NLF-space $\Gamma_c(\mathbf{e}^*\ Ver\ Eq_1(P, P))$ is identified *as linear space* with the space $\mathscr{X}^c_G(P)$ of all G-invariant vector fields ξ on P such that $p(supp(\xi))$ is a compact set. We stress that the topology on $\mathscr{X}^c_G(P)$ inherited from the one of $\mathscr{X}(P)$ is finer than that on $T_e(Aut(P)) = \Gamma_c(\mathbf{e}^*\ Ver\ Eq_1(P, P))$ if G is not compact.

Even the elements of the tangent space at $\varphi \in AutP$, that is of the NLF-space $\Gamma_c(s^*\ Ver\ Eq_1(P, P))$, where $s = \Gamma_\varphi$, can be represented by means of vector fields in $\mathscr{X}^c_G(P)$. Indeed, $T_\varphi\ AutP$ can be written as $T_eL_\varphi T_eAutP$ or $T_eR_\varphi T_e\ AutP$ where L_φ and R_φ are the left and the right translation by φ on $AutP$. For $\xi \in \mathscr{X}^c_G(P)$ and $u \in P$ with $p(u) = x$

$$(T_eL_\varphi I(\xi))(x) = T\eta(0_u, (T\varphi \circ \xi)(u), 0_e)$$

$$(T_e R_\varphi I(\xi))(x) = T\eta(0_u, (\xi \circ \varphi)(u), 0_e).$$

For every $\xi \in \mathscr{X}^c_G(P)$, we denote by L_ξ the left invariant vector field on $AutP$ defined by $L_\xi(\varphi) = T_e L_\varphi I(\xi)$, for $\varphi \in AutP$.

In NLF-manifolds, nothing is assured about existence of flow of a general vector field. However, it was proved in [6] that left invariant vector fields on *Diff M* admit a global flow. This allows to prove the Lie-algebraic (anti)-isomorphism of $\mathscr{L}(Diff\ M)$ with $\mathscr{X}_c(M)$ and to define the exponential map. By very similar arguments we can obtain analogous results in the case of *AutP*. In particular we prove that $\mathscr{L}(AutP)$ is (anti)-isomorphic to the Lie algebra $\mathscr{X}^c_G(P)$.

LEMMA 3.5. *Every left invariant vector field on AutP admits a global flow.*

Proof: First we remark that every $\xi \in \mathscr{X}^c_G(P)$ admits a global flow $Fl : P \times \mathbf{R} \to P$ satisfying $Fl(ug, t) = Fl(u, t)g$ for $u \in P$, $t \in \mathbf{R}$ and $g \in G$. This follows easily since $p(supp\ \xi)$ is compact and ξ is G-invariant. Since $p(supp\ \xi)$ is compact, there exists a compact subset K of M such that $Fl(u, t) = u$ if $p(u) \notin K$. Now construct the smooth map

$$a : M \times \mathbf{R} \to Eq(P, P'),\ a(x, t) = \eta(u, Fl(u, t), e)\ for\ p(u) = x,$$

and consider the map $\mathbf{R} \to \Gamma\ Eq_1(P, P)$, $t \to a_t$ with $a_t(x) = a(x, t)$, $x \in M$. For every $t \in \mathbf{R}$, the section a_t has support contained in K. By the arguments used in (4.4) of [6] one easily proves that the map $t \to a_t$ is a C^∞_c-map and results to be a one parameter subgroup of *AutP*. Therefore the map $\alpha : AutP \times \mathbf{R} \to AutP$, $\alpha(\varphi, t) = \varphi . a_t$ is C^∞_c.

We claim that α is the global flow of L_ξ. Actually, for $\varphi \in AutP$ and $t \in \mathbf{R}$,

$$\frac{d}{dt}\alpha(\varphi, t) = \frac{d}{dt}(\varphi . a_t) = TL_\varphi . \frac{d}{dt} a_t =$$

$$= TL_\varphi TL_{a_t} I(\xi) = L_\xi(\alpha(\varphi, t))$$

since for $x \in M$

$$\frac{d}{dt} a_t(x) = \frac{d}{dt}\eta(u, Fl(u, t), e) =$$

$$= T\eta\left(0_u, \frac{d}{dt} Fl(u, t), 0_e\right) = T\eta(0_u, \xi(Fl(u, t), 0_e) =$$

$$= T\eta(0_u, (T_u Fl(\cdot\ , t))\ \xi(u), 0_e) = TL_{a_t} I(\xi)(x) = L_\xi(a_t)(x).$$ ■

THEOREM 3.6. *The Lie algebra* $\mathcal{L}(AutP)$ *is anti-isomorphic to the Lie algebra* $\mathcal{X}^c_G(P)$.

Proof: We will show that the linear map I induces a Lie-anti-isomorphism. We have just to prove that for every $\xi, \xi' \in \mathcal{X}^c_G(P)$

$$[L_\xi, L_{\xi'}](e) = I([\xi', \xi]).$$

To compute $[L_\xi, L_{\xi'}](e)$ we use the formula

$$[L_\xi, L_{\xi'}] = \left.\frac{d}{dt}\right|_{t=0} \alpha_t^* L_{\xi'}$$

where α denotes the flow of L_ξ and, for every vector field X on $AutP$, α_t^* is given by

$$\alpha_t^* X = T\alpha_{-t} \circ X \circ \alpha_t.$$

By the argument in Lemma 3.5 we obtain

$$(L_{\xi'} \circ \alpha_t(e))(x) = L_{\xi'}(a_t)(x) = (T_e L_{a_t} I(\xi'))(x) =$$
$$= T\eta(0_u, (T_u Fl(\cdot\,, t)\, \xi'(u), 0_e)), \quad x \in M.$$

Hence by the above formula we have:

$$([L_\xi, L_{\xi'}](e))(x) = \left.\frac{d}{dt}\right|_{t=0} (T\alpha_{-t} \circ L_{\xi'} \circ \alpha_t)(e)(x) =$$

$$= T\eta\left(0_u, \left.\frac{d}{dt}\right|_{t=0} (T_u Fl(\cdot\,, t)) \circ \xi' \circ Fl(u, -t), 0_e\right) =$$

$$= T_\eta(0_u, [\xi', \xi](u), 0_e) = I([\xi', \xi])(x)$$

for $x \in M$. ■

The exponential mapping of $AutP$ is the mapping

$$Exp : \mathcal{X}^c_G(P) = T_e\, AutP \to AutP$$

which assigns to each vector field $\xi \in \mathcal{X}^c_G(P)$ the automorphism of P

$$Exp\, \xi = Fl(\xi)(\cdot\,, 1)$$

where $Fl(\xi) : P \times \mathbf{R} \to P$ is the global flow of ξ.

By arguments very similar to that used in 4.6 of [6], one obtains that Exp is a C^∞_c-map.

It is well known that the exponential map of the group of diffeomorphisms of a manifold fail to be a local diffeomorphism [1, 15] (see also [16]. Reasonably, we can expect the same feature of the exponential map of *Aut P*. Actually, we shall see in the last section that the image $Exp(\mathscr{X}^c_G(P))$ does not contain any open neighborhood of e.

We known by Theorem 3.1 that *Gau P* is a splitting normal Lie subgroup of *Aut P*. By means of Theorem 3.6 we give another natural realization of the Lie algebra of *Gau P*.

Recall the embedding $\iota : Gau\ P \to Aut\ P$ and denote by $\mathscr{X}^v(P)$ the Lie algebra of the vertical fields on P. Then the following theorem given the wanted characterization.

THEOREM 3.7. *The restriction of I to the Lie algebra* $\mathscr{X}^c_G(P) \cap \mathscr{X}^v(P)$ *is a Lie-anti-isomorphism with the Lie algebra* $(T\iota_*)\,(\Gamma_c P[g])$.

Proof: Remark that every vertical equivariant vector field ξ on P can be expressed as $\xi(u) = 0_u.h(u)$, where $h : P \to g$ is an equivariant smooth map. This realizes a bijection between $\Gamma_c P[g]$ and $\mathscr{X}^c_G(P) \cap \mathscr{X}^v(P)$. Consider now the embedding $\iota_* : Gau\ P \to Aut\ P$. The tangent map $T(\iota_*) = (T\iota)_*$ restricted to the tangent space at the identity $(T\iota_*) : \Gamma_c P\ [g] \to \Gamma(e^*\ Ver\ Eq_1(P, P)$ gives

$$(T\iota_*\ \sigma)(x) = T\eta(0_u, 0_u, h(u)) = T\eta(0_u, 0_u.h(u), 0_e) = I(\xi)$$

where $h : P \to g$ is the equivariant map defined by $\sigma(x) = [(u, h(u)]_G$ and $\xi(u) = 0_u\, h(u)$ belongs to $\mathscr{X}^c_G(P) \cap \mathscr{X}^v(P)$. Using the above remark one can easily obtain that the map I is onto $T\iota_*(\Gamma_c P[g])$. It is clear that I is a Lie-anti-isomorphism of $\mathscr{X}^c_G(P) \cap \mathscr{X}^v(P)$ with $(T\iota_*)(\Gamma_c P[g])$. ∎

Fixing a connection γ on P, we can explicitly construct a projection on the splitting subspace $I(\mathscr{X}^c_G(P) \cap \mathscr{X}^v(P))$ of $\Gamma_c(e^*\ Ver\ Eq_1(P, P))$. Let $V_\gamma : TP \to$ $\to Ver\ P$ be the projection on the vertical bundle defined by the connection γ. Then a continuous projection P_γ on $\Gamma_c(e^*\ Ver\ Eq_1(P, P))$ with range $I(\mathscr{X}^c_G(P) \cap \mathscr{X}^v(P))$ is defined by the following diagram

$$\begin{array}{ccc}
\mathscr{X}^c_G(P) & \xrightarrow{\ (V_T)_*\ } & \mathscr{X}^c_G(P) \cap \mathscr{X}^v(P) \\
\Big\downarrow I & & \Big\downarrow I \\
\Gamma_c(e^*\ Ver\ Eq_1(P, P) & \xrightarrow{\ P_\gamma\ } & \Gamma_c(e^*\ Ver\ Eq_1(P, P)).
\end{array}$$

4. Hom(*P*, *P*′), Emb$_G$ (*P*, *P*′) AND *Aut P* AS PRINCIPAL BUNDLES

In the above section we have seen that $Hom(P, P')$ and $Emb_G(P, P')$ are C_c^∞-manifolds and that $AutP$ is a NLF-Lie group. Here we prove that they are in fact principal bundles. In particular we prove that $AutP$ is an extension of $GauP$ in the category of NLF-Lie groups.

LEMMA 4.1. *The surjection* $\natural : Hom(P, P') \to C_\natural^\infty (M, M')$ *admits local sections at every* $f \in C_\natural^\infty (M, M')$.

Proof : We recall, that for every $f \in C_\natural^\infty (M, M')$, $Graph\ (f) = id_M \times f$ is a section of the fiber bundle $(M \times M', pr_1, M)$. Hence there exists a tubular neighborhood $W \subset M \times M'$ of its image, with vertical fibers, i.e. with projection $p_W = (Graph(f)) \circ pr_1$ (see (10.9) of [14]). Choose a connection on the principal bundle $P \times P'$ and use the induced connection γ on its associated bundle $(P \times P')\,[G]$ to lift curves in W : for a curve c in W, with $c(0) = w$ and a point $y \in Eq(P, P')$ over w, denote by $t \to P_T(c, t, y)$ the parallel transport of c starting from the point y.

For every $w \in W$ with foot point $p_W(w) = (x, f(x))$, construct the curve

$$c_w : \mathbf{R} \to W,\ c_w(t) = t.w$$

where the dot denotes the scalar multiplication defined in the fiber $W_{(x, f(x))}$ of the vector bundle W. Fix $s \in \Gamma\ Eq_1(P, P')$ such that $f = \omega_* s$ and construct a lift by

$$\mathscr{L} : W \to Eq(P, P'),\ \mathscr{L}(w) = P_T(c_w, 1, s(x))$$

if w has $(x, f(x))$ as foot point. The map $\mathscr{L}$ lifts globally the tubular neighborhood W around the image of the section s. We prove that $\mathscr{L}$ is a smooth section of the bundle $(Eq(P, P'), \alpha \times \omega, M \times M')$.

Clearly, $(\alpha \times \omega)\ \mathscr{L}(w) = c(1) = w$. To prove smoothness of $\mathscr{L}$, consider a trivializing neighborhood θ of $(Eq(P, P'), \alpha \times \omega, M \times M')$ with $w \in \theta \subset W$ and a restriction $\bar{c}_w$ of the curve c_w such that its image is contained in θ. The local expression of the parallel transport is a smooth map

$$\bar{p}_T(\bar{c}_w, \cdot, \cdot) : I \times G \to \theta \times G \qquad I \subset \mathbf{R}$$

and it is the «flow» of the time dependent vector field $\bar{\Gamma}(\partial \bar{c}_w / \partial t)$, on G, where $\bar{\Gamma} : T\theta \times G \to TG$ is the local expression of the connection γ. As the vector field $\bar{\Gamma}(\partial \bar{c}_w / \partial t)$, depends smoothly on the parameter w, its solutions, and hence P_T, depend smoothly on w.

Now we are ready to construct the wanted local section of the bundle

$(\Gamma Eq_1(P, P'), \natural, C^\infty_\natural(M, M'))$ in an open neighborhood of f. The subset U of all mappings h whose graph lies in W is open in the FD-topology by (3.2) and (4.7) of [14] and contains f.

Define

$$\sigma : U \to \Gamma\, Eq_1(P, P'), \sigma(h) = (\mathscr{L} \circ Graph)(h).$$

Clearly, σ is well defined. Moreover $(\natural \circ \sigma)(h) = \omega \circ \mathscr{L} \circ (id_M \times h) = h$. Finally, $\sigma = \mathscr{L}_* \circ Graph$ is a C^∞_c-map, since it is the composition of C^∞_c-maps. Therefore σ is the required local section at f. ∎

THEOREM 4.2. *Comp* : $Hom(P, P') \times AutP \to Hom(P, P')$ *is a* C^∞_c*-action and its restrictions to* $Hom(P, P') \times GauP$ *or* $Emb_G(P, P') \times AutP$ *are* C^∞_c *and free.*

Proof : *Comp* : $Hom(P, P') \times Aut\, P \to Hom(P, P')$ is a C^∞_c- action by Theorem 3.1 and Theorem 2.5. Moreover $\varphi \in Hom(P, P')$, $g \in AutP$, $\varphi \circ g = \varphi$ implies

$$\varphi^\natural \circ g^\natural = \varphi^\natural$$

and

$$g_x = \varphi^{-1}_{g^\natural(x)} \circ \varphi_x .$$

Therefore one is reduced to prove that the «lower action» is free. But this obvious in both case. ∎

The fibers of the surjection $\natural$ are exactly the orbits of the action of $GauP$. Actually, if $\varphi^\natural = \psi^\natural = f$, we define g by $g(x) = \varphi^{-1}{}_{f(x)}\psi_x$ for $x \in M$. Then $g \in GauP$ and $\varphi \circ g = \psi$. Moreover, g depends in a C^∞_c-way on φ and ψ by construction. Then the following theorem follows by Lemma 4.1 and Theorem 4.2.

THEOREM 4.3. *(*$Hom(P, P'), \natural, C^\infty_\natural(M, M'); GauP$*) is a* C^∞_c*- principal bundle.* ∎

COROLLARY 4.4. *(*$AutP, \natural, Diff_\natural\, M; GauP$*) is a* C^∞_c*-principal bundle.* ∎

We obtain therefore the expected exact seguence in the category of NLF-Lie groups

$$1 \to GauP \overset{\iota}{\to} AutP \overset{\natural}{\to} Diff_\natural\, M \to 1$$

in which, moreover, a «local splitting property» holds; that is we have exactly the same situation as in the category of ordinary Lie groups. The exponential maps commute with respect to corresponding exact sequence in the Lie algebras, i.e.

$$0 \to \mathscr{X}^c_G(P) \cap \mathscr{X}^v(P) \to \mathscr{X}^c_G(P) \to \mathscr{X}_c(M) \to 1$$

$$Exp \downarrow \qquad Exp \downarrow \qquad Exp \downarrow$$

$$1 \to GauP \to AutP \to Diff_\natural M \to 1 \ .$$

It is well known that $Exp : \mathscr{X}_c(M) \to DiffM$ does not contain any open neighborhood of e in its image [1, 15]. By the above diagram, we see that the same holds for $Exp : \mathscr{X}^c_G(P) \to AutP$. Actually, $U \subset Exp\, \mathscr{X}^c_G(P)$ would imply $\natural(U) \subset \subset Exp\ \mathscr{X}_c(M)$.

The interesting questions arise, whether the exact sequence above splits, i.e. a section of $\natural$ exists, which is a group homomorphism and whether $Diff_\natural M = DiffM$. The lifting of the entire *DiffM* seems to be strictly related to the naturality of the principal bundle. A related question is discussed in [17].

Since $Hom(P, P')$ is locally arcwise connected, we can use the following density theorem to obtain that every G-homomorphism is homotopic to a G-equivariant embedding, in the case of compact M.

THEOREM 4.5. *Let M be compact and $dimM' \geqslant 2\ \dim M + 1$. Then $Emb_G(P, P')$ is dense in $Hom(P, P')$.*

Proof. Under our assumptions $Emb(M, M')$ is dense in $C^\infty(M, M')$ by (2.13) of [18] and therefore $Emb_\natural(M, M')$ is dense in $C^\infty_\natural(M, M')$. Then the assertion follows immediately from the local triviality of the principal bundles $(Hom(P, P'), \natural, C^\infty_\natural(M, M'); GauP)$ and $(Emb_G(P, P'), \natural, Emb_\natural(M, M'); GauP)$. ■

Finally, coming to $Emb_G(P, P')$, we recall (see (13) of [14]) that the action of *DiffM* on $Emb(M, M')$ gives a C^∞_c- principal bundle $(Emb(M, M'), u, U; DiffM)$ with base the NLF-manifold U of the equivalence classes of embeddings, where the equivalence is

$$f \sim h \quad \text{if there exists } g \in Diff\,M,\ h = f \circ g,\ f, h \in Emb\ (M, M').$$

As a trivial consequence of this result and of 3), Theorem 2.4, we obtain that $(Emb_\natural(M, M'), u, U_\natural, Diff_\natural(M))$ is a C^∞_c- principal bundle, where $U_\natural$ is the set of equivalence classes of embeddings in $Emb_\natural(M, M')$, with respect to the above equivalence relation.

Finally, by Theorem 4.3 and the above quoted results we have.

THEOREM 4.6. *$(Emb_G(P, P'), u \circ \natural, U_\natural; AutP)$ is a C^∞_β- principal bundle.* ■

An interesting question would arise rather naturally at this point, that is to investigate the topology of the above principal bundles in the relevant case in which P' is taken to be an n-universal bundle EG. If M is not compact, $Hom(P, EG)$ may even be not connected. In the case of compact M one can use the n-connectness of EG to prove that $\Gamma(P[EG]) = Hom\ (P, EG)$ is a $GauP$ classifying bundle.

The classification of $GauP$ (or $AutP$)-principal bundles and related questions will be discussed in a separate paper.

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